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On Stability and Transition in Three-Dimensional Flows

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The usefulness of the e^n method for predicting transition in two-dimensional and axially symmetric flows is well established. In order to extend the method to three-dimensional parallel shear flows, it is first necessary to establish a relationship between α and β, the complex wave numbers in two perpendicular directions in the plane of flow. We suggest that this may be done by making use of group velocity concepts which lead to the requirement $\partial\alpha/\partial\beta = -\tan\phi$, where ϕ is real and denotes the direction of propagation of centered disturbances. As a paradigm of this approach, the rotating disk is studied. It is established that the critical Reynolds number is 176, that the principal disturbances to the laminar flow travel outwards and at an angle $\simeq -8$ deg to the direction of motion of the disk, while the appropriate value of n is ~ 20. The observed direction of propagation of disturbance is at $\simeq -13$ deg to the direction of motion of the disk. The generally accepted value of n is ~ 9, much less than that found here.

Introduction

THE prediction of transition in a boundary layer is a problem of central importance to the computation of flows past bodies, but also one of great difficulty, depending on a large number of flow characteristics in ways which are far from being fully understood. The most successful of the empirical methods available at present is the "e^n" method introduced by Smith and Gamberoni[1] who took $n=9$ and Van Ingen[2] who took $n=7$ or 8. Recent reviews of this method have been written by Berger and Aroesty[3] and Mack.[4] It appears that the method works reasonably well for incompressible two-dimensional and axisymmetrical flows past smooth bodies, provided the mainstream turbulence is low. It is based on the Orr-Sommerfeld theory of small disturbances in a parallel shear flow, and transition is defined to occur when a disturbance, supposed to originate at its critical Reynolds number, has first been amplified by a factor e^n ($\doteqdot 8000$ if $n=9$).

A broad interpretation of this empirical result is that nonlinear effects are negligible in the early stages of the flow downstream of the critical Reynolds number and that when they do become significant, there is a rapid collapse of the ordered flow. Theories leading to the Stuart-Landau equation (Stuart,[5] reviewed by Stewartson[6]), lend support to the reasonableness of this idea. They also indicate that the method is unlikely to be applicable when the flow is strongly subcritically unstable; i.e., when the real part of the Landau constant is large and positive. This is the situation with plane Poiseuille flow[7] and then the breakdown of the ordered flow to turbulence occurs through mechanisms that have little to do with the Orr-Sommerfeld equation.[8] For Blasius flow, a calculation of the Landau constant has been carried out by Itoh[9,10]; at the critical Reynolds number R_c, the real part is virtually zero and positive near the upper branch of the neutral curve. The interpretation of his findings is rather opaque[11,12] in view of the parallel-shear assumption and the strict inapplicability of the theory when $R \neq R_c$, but it seems that the nonlinear terms do not become immediately important as R increases through R_c, in qualitative agreement with the method. The experimental studies of Gaster and Grant[13] also support the view that nonlinear terms are not important just downstream of R_c, provided that the mainstream turbulence is low and the body smooth.

In view of the practical success achieved with this method, it is of interest to investigate its application to three-dimensional situations. A convenient starting point is the boundary layer on a rotating disk which has some symmetry properties even though the direction of flow varies across it. The first experimental studies were carried out by Smith[14] and repeated in more detail by Gregory et al.[15] when the ambient fluid is quiescent. They observed the first signs of instability at $R \simeq 430$ and transition at $R \simeq 530$ where the Reynolds number $R = a(\Omega/\nu)^{1/2}$, a being the distance from the axis of rotation, Ω the angular velocity of the disk, and ν the kinematic viscosity. Between these two Reynolds numbers a china clay technique revealed equiangular spirals on the disk inclined at an angle $\simeq 167$ deg to its direction of motion. An explanation was offered by Stuart, but, since it was based on inviscid theory, it must be regarded as partial even if highly suggestive. Chin and Litt[16] later repeated these studies, finding a slightly wider band $410 < R < 590$ for transition, and Brown[17] estimated the critical Reynolds number at $R \simeq 173$. In the related problem of Ekman boundary layers an ordered flow for a range of values of $R > R_c$ has also been observed.[18,19] In addition, Iooss et al.[20] have demonstrated that the instability of the basic flow is supercritical. By analogy, it is quite possible that the instability is supercritical in the rotating disk problem too; hence, the Orr-Sommerfeld theory might be of use in estimating an n for the e^n method. Some discussions of the temporal and spatial eigenvalues have recently been given by Mack[4] and Cebeci and Keller,[21] partially with this in mind.

In this paper we shall make use of a modification of a method of Cebeci and Keller[21] (see also Ref. 22) for computing the eigenvalues of the Orr-Sommerfeld equation to investigate the spatial evolution of linear disturbances. The method is especially convenient for such a study because of the constraints that must be exerted on the wave numbers of

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the oscillation. As a preliminary, consider temporal eigenvalues that are important if we wish to examine the evolution in time of an initial localized disturbance. Then we take a double Fourier transform with respect to coordinates x,y on the surface of the disk and study the time evolution of the resulting simpler equation. Formally, we assume that a typical dependent variable $\bar{Q}$ is of the form

$$Q(z)\exp[i(\alpha x+\beta y+\omega t)] \tag{1}$$

where α,β are arbitrary real numbers, and ω is found from the Orr-Sommerfeld equation. Clearly, if $\omega_i>0$ where $\omega=\omega_r+i\omega_i$, then the basic flow is unstable to linear disturbances. However, for the transition problem, we need to consider how a disturbance, originating at a point P evolves as it moves downstream with the fluid. Hence, we should take ω real corresponding to the Fourier transform of a finite disturbance at the origin and compute, in some sense, the values of α and β as functions of ω and R. By allowing R to be a slowly varying function of position, we may then estimate the amplitude of the disturbance at any station and so arrive at a criterion for transition of the e^n kind. Gaster[23] has found that the growth properties of small disturbances in a laminar boundary layer are described quite well by this method. We are now faced with a difficulty because the requirement of a nontrivial solution of the Orr-Sommerfeld equation only provides two relations connecting the numbers α,β,ω (all of which may be complex) and R. Of these, R and ω are prescribed and, therefore two new relations connecting α and β must be given before the solution can be completed. These follow by adapting the notions of group velocity, well known in connection with dispersive systems and two-dimensional shear flows, to general planar-shear flows.[24] We shall find that for a disturbance from the origin to penetrate to specified large values of x and y we must have $x(\partial\alpha/\partial\beta)+y=0$ giving two conditions connecting α and β. For given values of ω and R, we may now use the Orr-Sommerfeld equation to obtain admissible values of α and β. These are not necessarily unique, but we shall assume that they are, or, if not, that we may select the most significant. Hence, for given ω,R and direction ϕ of propagation, we may determine whether it is stable and, if not, its rate of growth. Finally, by varying R we may estimate its amplification from the neutral point to transition which is needed for an application of the e^n method.

For a given value of ω, a neutral curve can be found on which the growth rate is zero ($\alpha_i+\beta_i\tan\phi=0$, $y=x\tan\phi$) and along which R,α,β and ϕ all vary. At the minimum value $R(\omega)$ of R on this curve, $\alpha_i=\beta_i=0$; it is convenient to define the locus of $R(\omega)$ as ω varies on the absolute neutral curve [hereafter called the zarf (lit. envelope, Turkish)] and use it as starting points for the calculation of n. We note that this curve is also the zarf for temporal disturbances defined as the envelope of all temporal neutral curves.

Basic Equation

We choose orthogonal Cartesian axes $OXYz$ fixed relative to the rotating disk with Oz along the axis of rotation and origin O in the plane of the disk. Let aX, aY denote distances along the two axes in the disk plane and $aR^{-1}z$ distance perpendicular to the disk, where a is a characteristic length and $R=a(\Omega/\nu)^{1/2}$. Further, let Ωau, Ωav, ΩawR^{-1} be the corresponding components of velocity of the fluid. Then it is well known (see Ref. 25 for a recent review) that if the angular velocity of the ambient fluid is $\sigma\Omega$, there is a steady-state solution of the Navier-Stokes equation of the form

$$u=XF(z)-YG(z) \quad v=YF(z)+XG(z) \quad w=H(z) \tag{2}$$

satisfying the no-slip condition on the disk and the ambient conditions as $z\to\infty$, provided σ does not lie in the range $-6.2<\sigma+1<-0.70$. The differential equations satisfied by F, G, H are:

$$2F+H'=0, \qquad F''-HF'-F^2+(G+1)^2=(\sigma+1)^2,$$
$$G''-HG'-2F(G+1)=0 \tag{3}$$

Here primes denote differentiation with respect to z. Our main interest in this paper is when $\sigma=-1$, but there are some studies when σ is small that are relevant.

We are interested in perturbations about this basic solution which satisfy the linearized form of the Navier-Stokes equations and are centered, without loss of generality, at the line $P(1,0,z)$ of the disk and with scale lengths in all directions of the same order of magnitude as the thickness aR^{-1} of the shear layer. Specifically, we write

$$X=1+xR^{-1}, \quad Y=yR^{-1}, \quad w=H(z)+R\tilde{w}(x,y,z,t)$$
$$u=XF(z)-YG(z)+\tilde{u}(x,y,z,t)$$
$$v=YF(z)+XG(z)+\tilde{v}(x,y,z,t) \tag{4}$$

where $t/(R\Omega)$ is time. Then the equations governing the linear perturbations $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ are:

$$R\left[\frac{\partial\tilde{u}}{\partial t}+F\frac{\partial\tilde{u}}{\partial x}+G\frac{\partial\tilde{u}}{\partial y}+\tilde{w}\frac{dF}{dz}-\frac{\partial\tilde{\pi}}{\partial x}\right]-\nabla^2 u$$
$$=-(xF-yG)\frac{\partial\tilde{u}}{\partial x}-(yF+xG)\frac{\partial\tilde{u}}{\partial y}-H\frac{\partial\tilde{u}}{\partial z}$$
$$-\tilde{u}F+\tilde{v}G+2\tilde{v}-\tilde{w}\left(x\frac{dF}{dz}-y\frac{dG}{dz}\right) \tag{5a}$$

$$R\left[\frac{\partial\tilde{v}}{\partial t}+F\frac{\partial\tilde{v}}{\partial x}+G\frac{\partial\tilde{v}}{\partial y}+\tilde{w}\frac{dG}{dz}-\frac{\partial\tilde{\pi}}{\partial y}\right]-\nabla^2 v$$
$$=-(xF-yG)\frac{\partial\tilde{v}}{\partial x}-(yF+xG)\frac{\partial\tilde{v}}{\partial y}-H\frac{\partial\tilde{v}}{\partial z}$$
$$-\tilde{u}G-\tilde{v}F-2\tilde{u}-\tilde{w}\left(y\frac{dF}{dz}+x\frac{dG}{dz}\right) \tag{5b}$$

$$R\left[\frac{\partial\tilde{w}}{\partial t}+F\frac{\partial\tilde{w}}{\partial x}+G\frac{\partial\tilde{w}}{\partial y}-\frac{\partial\tilde{\pi}}{\partial z}\right]-\nabla^2\tilde{w}$$
$$=-(xF-yG)\frac{\partial\tilde{w}}{\partial x}-[yF+xG]\frac{\partial\tilde{w}}{\partial y}-H\frac{\partial\tilde{w}}{\partial z}-\tilde{w}\frac{dH}{dz} \tag{5c}$$

$$\frac{\partial\tilde{u}}{\partial x}+\frac{\partial\tilde{v}}{\partial y}+\frac{\partial\tilde{w}}{\partial z}=0 \tag{5d}$$

where $\tilde{\pi}$ is a reduced pressure and $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ vanish both at $z=0$ and at $z=\infty$.

In general, these equations cannot be solved by separation of variables, as implied by Eq. (1), and recourse must be made to an expansion in descending powers of R, the leading term being found by neglecting the right-hand sides of Eq. (5). This difficulty and the procedure it compels are familiar in the stability theory of the Blasius boundary layer. Attempts to estimate the error have led to some controversy[26-31] and the experimental evidence is not consistent.[32,33] There is something to be said for Gaster's conclusion[28] that the next term in the expansion, while improving the agreement, is still insufficient to account for the discrepancy in the value of the critical Reynolds number between Orr-Sommerfeld theory, based on the leading term and experiment.

The stability of the Ekman layer, obtained from Eq. (3) by taking $0<\sigma\ll 1$, is one shear flow for which an exact theory is possible. We note that now F,G,H are proportional to σ and so we write $t=t\sigma$, $R_E=R\sigma$ and let $\sigma\to 0$, $R\to\infty$ holding R_E constant. The governing equations, Eqs. (5), simplify in that the right-hand sides reduce to the Coriolis terms; namely $+2\tilde{v}$, $-2\tilde{u}$, 0, 0, respectively. Separation of variables is feasible and so a direct check on the expansion procedure is possible. Theoretical studies have been made by Lilly,[34] Faller and Kaylor,[35] and Brunsvold and Vest[36] and experiments have been performed by Faller[37] and Tatro and Möllo-Christensen[18] which largely confirm the full theory in the context of more general geostrophic flows in the ambient fluid than the pure rotation just assumed. For $R_E>150$ the Orr-Sommerfeld theory and the full theory are in reasonable agreement, but there are important discrepancies at lower values of R_E. The principal reason seems to be that a new class of instabilities occurs with the full equations for $R_E<100$, intimately linked with the Coriolis forces which leads to a diminution of the critical value of R_E from 93 to 55. Although this type of instability merges with the other as R_E increases, it may have some relation to a long-wave instability at large values of R (described by Bodonyi[38]) when $\sigma+1<0$.

It is clear from this discussion that further consideration of the effect of the right-hand sides of Eq. (5) is needed before we can fully assess the errors made by neglecting them. There is, however, some justification for believing that if $R>200$, they may safely be neglected and we shall proceed accordingly. The governing equations may now be solved by separation of variables and so we write a typical dependent variable $\tilde{Q}$ as

$$\tilde{Q}=Q(z)\exp[i(\alpha x+\beta y-\omega t)] \tag{6}$$

where α,β,ω are constants and Q is to be found. Further, let

$$\tilde{w}=-i\Psi(z)\exp[i(\alpha x+\beta y-\omega t)] \tag{7}$$

and it then easily follows[15] that

$$\Psi''''-2(\alpha^2+\beta^2)\Psi''+(\alpha^2+\beta^2)^2\Psi-iR[(\alpha F+\beta G-\omega)$$

$$(\Psi''-(\alpha^2+\beta^2)\Psi)-(\alpha F''+\beta G'')\Psi]=0 \tag{8a}$$

with

$$\Psi(0)=\Psi'(0)=0 \tag{8b}$$

$$\Psi(\infty)=\Psi'(\infty)=0 \tag{8c}$$

This equation is, of course, the Orr-Sommerfeld equation of the rotating disk problem. We shall now investigate the properties of the nontrivial solutions, particularly the eigenvalues relevant to the prediction of transition.

Spatial Eigenvalues and Group Velocity

We may expect that for general values of α,β a nontrivial solution of Eqs. (8) can be found for at least one value of ω, provided $R>R_0$, where R_0 is a positive number and probably quite small. Second, the existence of solutions to Eqs. (8) is not central to the mathematical problem of describing the evolution of small disturbances to the steady rotational flow. The central role is provided by the continuous spectrum.[39]

Suppose, for example, that a disturbance is made in the fluid in the neighborhood of the point P of the disk at $t=0$. Let us examine its evolution on the assumptions that the Reynolds number R is constant over the region of interest (all finite x,y) and the right-hand sides of Eq. (4) may be neglected. These assumptions are justified only in the limit $R\to\infty$ and are only approximately true in the specific problem of the rotating disk, for which $200<R<600$. Hence, the application of an otherwise precise theory to this problem is subject to an error which we expect to be small.

A double Fourier transform in (x,y) is appropriate which leads us to Eqs. (8) together with a forcing term on the right-hand side due to the initial disturbance. The solution for Ψ, regarded as a function of α and β, has branch points when $\omega+\beta=-i(\alpha^2+\beta^2)R^{-1}$, corresponding to the continuous spectrum. These may be interpreted as generalizations of the eigenvalues of the heat conduction equation to which the perturbation equations reduce as $z\to\infty$. It may (or it may not) also have poles on the real axes of α,β and these correspond to the temporal eigenvalues of Eqs. (8). In all known examples to date the continuous spectrum consists of stable eigensolutions, i.e., $\omega_i\leq 0$, and so we must look to Eqs. (8) for instability. Other branch points or poles arising from the forcing disturbance may be avoided by choice of contours in the inverse Fourier transform and hence are of no interest to the initial value problem. Thus we see that the critical Reynolds number R_c for this problem is that value of R such that, for $R<R_c$, $\omega_i<0$ for all real α,β, while if $R>R_c$ it is possible to choose at least one combination of real α,β so that $\omega_i>0$. We may expect that if $R>R_c$, there is a closed curve $R(\alpha,\beta)=0$ in the α,β plane such that $\omega_i>0$ inside and $\omega_i<0$ outside. Further, according to Rayleigh's criterion, the basic flow is unstable to some inviscid disturbances; hence, this curve is finite when R is large. The determination of R_c and $R(\alpha,\beta)$ is straightforward once an appropriate numerical scheme is available, such as that used by Cebeci and Keller,[21] since the prescription of α,β,R determines ω. Any nonuniqueness may be handled by requiring that the ω_i chosen is the maximum among those found.

The initial value problem is, however, not directly relevant to the transition problem, because we wish to follow a disturbance as it is carried along by the fluid away from its origin P and especially out of its neighborhood, i.e., to values of X significantly greater than unity. This means that we must examine what happens to it as x,y increase to large values. Now the disturbance is bounded at $x=y=0$ for all time and may be regarded as zero for sufficiently large and negative values of t. It may, therefore, be analyzed by means of a Fourier transform with respect to t so that ω in Eq. (6) may be regarded as real. Again, Fourier transforms with respect to α or β are appropriate but no longer do we need to restrict attention to α,β real. As before, we are led to the solution of Eqs. (8) with a right-hand side due to conditions at $x=y=0$. The appropriate eigenvalues have a continuous spectrum where $\alpha^2+\beta^2=iR(\omega+\beta)$ and another family corresponding to the solutions of Eqs. (8) without the right-hand side. Now, however, we have a considerable uncertainty because the prescription of ω and R is not sufficient to fix α; β must also be prescribed.

The crucial question which concerns us is the strength of the disturbance at the point (x,y,z) at time t where $x^2+y^2\gg 1$. In order to find this, the concept of group velocity is needed and this enables us to write down a constraint on α,β subject to which the Orr-Sommerfeld equation has effectively a unique solution when ω,R are given.

It is well known that the group velocity of a wave packet in a dispersive medium gives its speed of propagation through space (see Ref. 40 for a recent account). There is, however, little published literature for dissipative media; Benjamin[41] has used the notion to study instabilities in thin films and Criminale and Kovasznay[42] and Gaster and Davey[43] in plane parallel flows. The extension to more general parallel flows such as we are considering here is straightforward. Suppose an oscillatory disturbance with reduced period $2\pi/\omega$ is generated at the origin. Then the disturbance at large finite values of x,y may reasonably be assumed to be of the form

$$\tilde{Q}(x,y,z;t)=e^{-i\omega t}\int_c Q(\alpha,\beta;\omega,z)e^{i(\alpha x+\beta y)}\,d\beta \tag{9}$$

where C is a contour in the complex plane of β extending to ∞ in either direction, Q is a determinate function of $(\alpha,\beta;\omega,z)$ whose properties are such that the integral converges, and α is a function of β,ω found by solving the Orr-Sommerfeld Eqs. (8). There are several important assumptions made in this statement, e.g., that solutions of Eqs. (8) exist for all β and the continuous spectrum may be neglected when $x^2+y^2 \gg 1$, but there seems to be no reason to neglect its correctness for unstable situations. We now write

$$y = x\tan\phi \qquad |\phi| \leq \pi/2 \tag{10}$$

where ϕ is a constant and look for the dominant contribution to $\tilde{Q}$ as $x \to \infty$. This arises from the saddle point of $\alpha x + \beta y$, regarded as a function of β, which occurs when

$$\frac{\partial\alpha}{\partial\beta}\bigg|_{\omega,R} + \tan\phi = 0 \tag{11}$$

ω, R being held constant during the differentiation.

Then

$$\tilde{Q} \sim \frac{Q_0}{|x|^{1/2}} \exp[-i\omega t + i(\alpha x + \beta y)] \tag{12}$$

as $x \to \infty$ where Q_0 is a finite function of z proportional to or derived from the eigenfunction Ψ, evaluated at the given values of ω, R, and subject to Eq. (11), and also depending on the specific component of the disturbance being considered. The constraints on α,β defined by Eq. (11) are sufficient to define α,β uniquely in Eq. (12) or, if not, to reduce the nonuniqueness to a discrete set of values. If there is a choice, the appropriate solution is the one for which

$$-(\alpha_i + \beta_i \tan\phi) = \Gamma \tag{13}$$

is a maximum so that it grows fastest with x. It should be noted that if $\Gamma \leq 0$, the above argument is of doubtful relevance since the continuous spectrum may be of more importance. There is a possibility that Eqs. (8) have no solutions satisfying Eq. (11). None has been found for $\phi > 0$, for example. In that event, the disturbance propagating in that direction is much weaker and may indeed die out, even though the basic flow is unstable.

A further argument for reducing the nonuniqueness is to assume that the time-dependence of the disturbance at P is of the form

$$\exp[-i\omega t - \epsilon^2 t^2] \tag{14}$$

where $\epsilon \ll 1$, the earlier one being obtained from Eq. (14) in the limit $\epsilon \to 0$. We now inquire the time at which the disturbance at (x,y,z) reaches a maximum when $x^2+y^2 \gg 1$. The appropriate generalization of Eq. (9) is:

$$\tilde{Q} = \frac{1}{2\sqrt{\pi}\epsilon}\int_{-\infty}^{\infty} d\Omega \int_c \exp\{-i\Omega t - [(\Omega-\omega)^2/4\epsilon^2]$$

$$\times Q(\alpha,\beta,z)\}\exp[i\alpha x + i\beta y]d\beta \tag{15}$$

The integrand has stationary phase when

$$ix\delta\alpha + iy\delta\beta - \frac{\Omega-\omega}{2\epsilon^2} - it\delta\Omega = 0 \tag{16}$$

for all small variations in β,Ω and consequent variation in α. Now suppose x,y,t are all large but that $x\epsilon^2$, $|y|\epsilon^2$, and $t\epsilon^2$ are small. Then at the stationary point, $\omega \simeq \Omega$ so that Eq. (11) must still hold. Further the contribution to the integral from the neighborhood of the stationary point has a temporal dependence dominated by the factor

$$\exp - \epsilon^2\left(\frac{\partial\alpha}{\partial\Omega}x - t\right)^2 \tag{17}$$

where $\partial\alpha/\partial\Omega$ is evaluated at $\Omega=\omega$ and at those values of α,β such that Eq. (11) holds and Eqs. (8) have a nontrivial solution. Thus, the disturbance at (x,y,z) has a maximum when

$$t = xRe(\partial\alpha/\partial\Omega) \tag{18}$$

and it follows that

$$Re(\partial\alpha/\partial\Omega) > 0 \tag{19}$$

for a physically acceptable solution (otherwise t decreases as x increases). This result remains true in the limit $\epsilon \to 0$.

If the condition $x\epsilon^2 \ll 1$ does not hold, then the stationary point of the exponent, defined by Eq. (16), moves away from $\Omega=\omega$ and, for large enough x, to the complex frequency such that

$$x\frac{\partial\alpha}{\partial\Omega} = t \qquad x\frac{\partial\alpha}{\partial\beta} + y = 0 \tag{20}$$

which has nothing to do with the imposed frequency ω.

Growth of Disturbances

Let us now consider the growth of a disturbance with frequency ω and direction of propagation inclined at an angle ϕ to the outward drawn radius on the assumption that the linearized theory holds. Given that when x,y are large, a typical element $\tilde{Q}$ of the motion takes the form Eq. (12), where we regard Q_0 as the effective initial disturbance at P, we now consider how it changes when X increases significantly from unity. There is, of course, no precise theory in these circumstances, partly because R is finite, thus an asymptotic approach is strictly unjustified. In physical terms, we may think of the disturbance as following an equiangular spiral inclined at the angle ϕ to the radial direction, as exemplified by the streaks observed in the experiments of Smith[14] and Gregory et al.[15] The variation in the amplitude of Eq. (12) with X may be approximated if $-(\alpha_i+\beta_i \tan\phi)x - \frac{1}{2}\log x$ is replaced by

$$-\int\left[(\alpha_i+\beta_i\tan\phi)R + \frac{1}{2(X-1)}\right]dx \tag{21}$$

and so, when X changes significantly,

$$|Q| \sim \frac{R_P^{1/2}|Q_0|}{(r-r_0)^{1/2}}\exp - \int_{R_P}^{R}(\alpha_i+\beta_i\tan\phi)dR \tag{22}$$

where $r=aX$, $r_0=a$ and $R_P(\omega,\phi)$ is the Reynolds number at the origin P of the disturbance. It is noted that in his study of the evolution of Tollmein-Schlichting waves in a laminar boundary-layer boundary layer, Gaster[20] found that the growth in their amplitude has a form similar to Eq. (22). The main difference occurs in the algebraic factor which, due to the boundary-layer growth, has the exponent ¼.

The principle underlying the e^n method for transition may be applied to the present problem as follows. Consider a disturbance with period $2\pi/\Omega R_P\omega_P$, originating a point P of the disk where R_P is such that the disturbance is initially neutral, i.e., $\alpha_i+\beta_i\tan\phi_P=0$ and is traveling outboard in the direction ϕ_P. We now follow this disturbance through increasing values of X and, when transition is reached, we determine the logarithm of the factor by which the amplitude has increased. Denoting this number by N, we next look for the maximum value n of N as ω_P, R_P vary, as well as the direction ϕ of propagation for $R>R_P$; this is the relevant n of the e^n method. For the flows mentioned in the introduction $n \simeq 9$ and one of the purposes of this paper is to investigate how it changes in this fully three-dimensional boundary layer.

There is no mathematically precise way of finding N at present, but the obvious approximate method is to define

$$N=-\int_{R_P}^{R_T}(\alpha_i+\beta_i\tan\phi)\mathrm{d}R \tag{23}$$

and to use the Orr-Sommerfeld theory to determine α_i, β_i at any $R>R_P$ for a real frequency $\omega=\omega_P R_P/R$ and the direction ϕ. The upper limit of integration in Eq. (23) is the Reynolds number chosen for the start of transition.

Zarf

The effort needed to find an n as outlined in the previous section is considerable. First it is necessary to devise a numerical scheme that enables us to solve the Orr-Sommerfeld equation subject to the requirement that $\partial\alpha/\partial\beta$ is real and also to follow that solution as ω,R vary. We have modified the scheme used by Cebeci and Keller[21] to perform these tasks, but it is inappropriate to describe the method here since we wish to emphasize the physical and engineering aspects of the problem and the new program is somewhat involved. It is hoped to describe the method in a separate paper.

In addition, the search over all ω_P, $\phi_P>0$ is a large undertaking and we restrict attention here to a class of initially neutral disturbances which we believe are the most important and are defined as follows. For a fixed real ω and $\partial\alpha/\partial\beta$ real and equal to p, a neutral curve in six-dimensional space connecting, α,β,R,p can be drawn. The projection of the curve on the R,β_r plane, for example, has a minimum value $R(\omega)$ and, at this point, $\alpha_i=\beta_i=0$. The reason is that for a small perturbation about this point, $\delta R=0$ and, since $\alpha_i=p\beta_i$ on the curve, $\beta_i\delta p=0$, i.e., $\beta_i=0$.

We may now define the zarf as the locus of points for which

$$\alpha_i=\beta_i=0 \qquad \partial\alpha/\partial\beta \quad \text{is real} \tag{24}$$

and is the locus of this minimum Reynolds number $R(\omega)$. It is plausible to expect that disturbances originating on the zarf are the most important from the point of view of transition. This zarf is also relevant to temporal eigenvalues and again, taking its projection on the R,β_r plane, we may define it as follows. Let $(\alpha_l, \beta_l, \phi_l, R_p)$ be a point on the curve. Then for all α,β and $R<R_P$, $\omega_i<0$ so that the disturbances are damped; for $R=R_p$, $\omega_i<0$ except when $\alpha_r=\alpha_l$, $\beta_r=\beta_l$; for $R>R_p$, there is at least one value of α_r such that $\omega_i>0$. It is tacitly assumed here that all hyperplanes on which α_r, β_r, or ω_r are constant, intersect the curve in at most one point but generalization is straightforward. In order to show that this definition also leads to Eq. (24), we write an eigenvalue of the Orr-Sommerfeld equation in the form

$$\omega=F(\alpha,\beta,R) \tag{25}$$

and observe that if $\omega_i=0$, $F_i(\alpha,\beta,R)=0$. Hence, since R is stationary at a point of the zarf $\partial F_i/\partial x=\partial F_i/\partial\beta=0$, it follows that $\partial\omega/\partial\alpha$ and $\partial\omega/\partial\beta$ are real and we are led immediately to Eq. (24).

Results and Discussion

The zarf (Fig. 1) was first obtained together with neutral curves for various values of ω (Fig. 2) by using our general program for solving the Orr-Sommerfeld equation. We found that the critical Reynolds number R_c, above which at least one temporal mode is amplified and below which none is amplified is approximately 175.6, close to Brown's result[17]; the associated parameters are $\alpha_c=0.348$, $\beta_c=0.031$, $\omega_c=0.017$, and $\partial\alpha/\partial\beta=6.95$. Thus, the critical oscillation has a phase velocity approximately in the outward radial direction, but a group velocity directed almost tangentially and in the direction of Y decreasing. This property is unusual in dissipative systems, but is well known in the theory of general dispersive systems, a good example being provided by Rossby waves.[44] Further, on the lower branch of the zarf, the direction of the phase velocity soon starts pointing radially inward. This property of the phase velocity is, however, of no great significance to the transition problem, since the physical

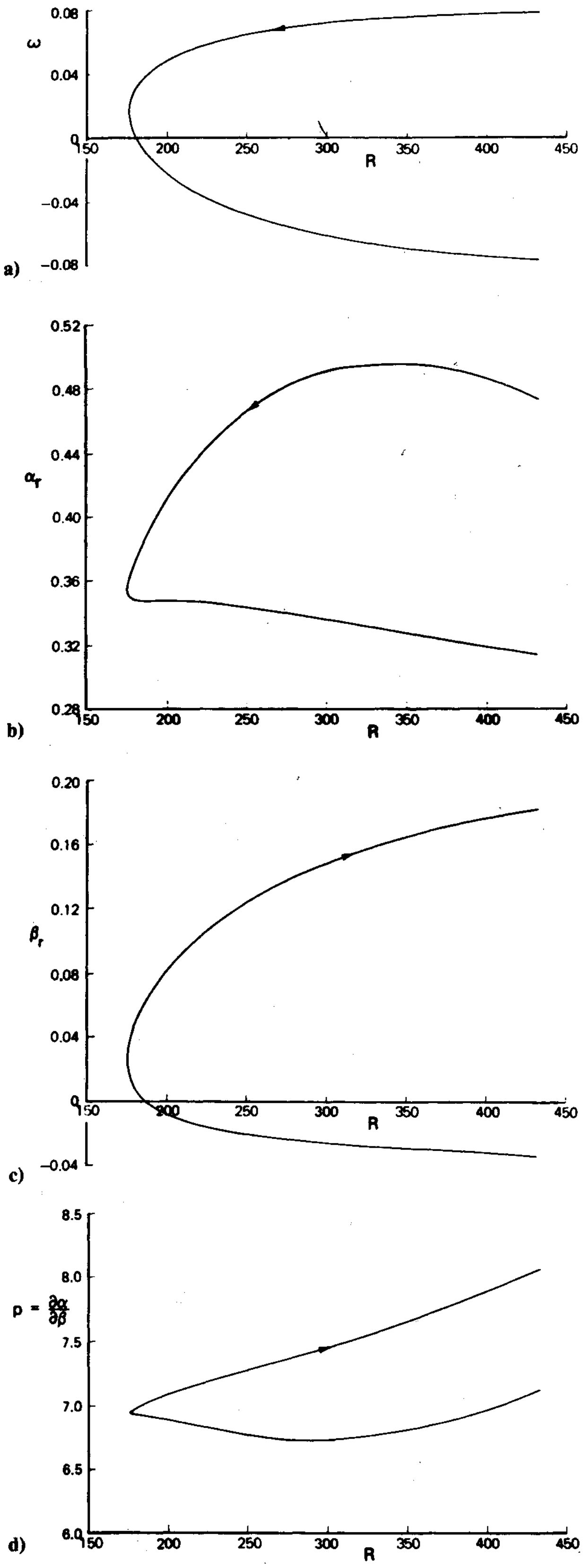

Fig. 1 Variation of ω, α_r, β_r and $p=\partial\alpha/\partial\beta$ with R on the zarf ($\alpha_i=\beta_i=p_i=0$).

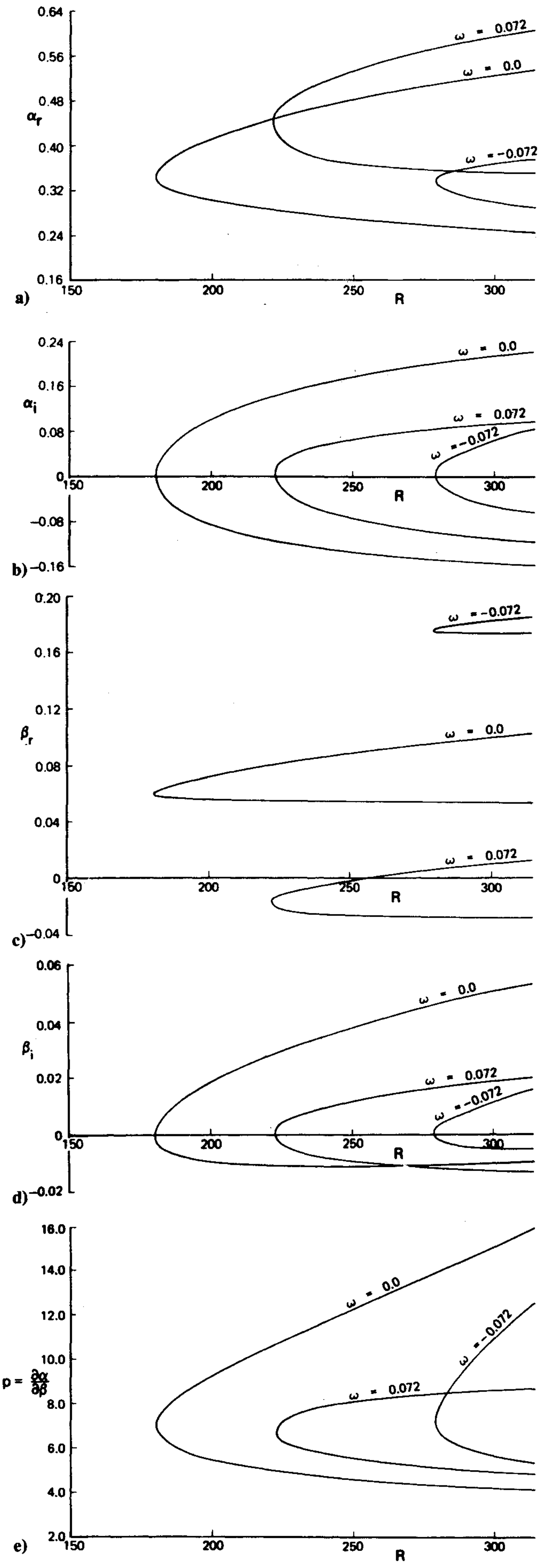

Fig. 2 Variation of α_r, α_i, β_r, β_i, p with R on the spatial neutral curves for $\omega = 0.072$, 0, and -0.072.

situation which produces it, i.e., a uniform wave train over the whole neighborhood of P and existing for all time, cannot occur as a disturbance of the steady flow due to the rotating disk. Instead, we must consider centered disturbances that travel away from P with the group velocity. The implication of Fig. 1d is that such disturbances begin by traveling outwards but at a very small, almost constant, angle (~ 8 deg) to the negative of the direction of motion of the disk, i.e., they are left behind by the rotating disk. This might have been expected on general physical grounds as, relative to the disk, the fluid is mainly moving backward in the tangential direction only having a smallish mean radial velocity outward and the disturbances are broadly carried by the fluid. Three of the neutral curves are displayed in Fig. 2 and the special features that α_i and β_i are no longer zero should be noted ($\alpha_i = p\beta_j$). Further, the value of p is always >4, indicating that these disturbances also travel almost parallel to the negative tangential direction.

Next we compute the value of n at transition according to Eq. (23), starting at the zarf and holding ϕ constant at the value it takes there. Both of these decisions on procedure need commenting on. The reader is first reminded that there is no complete logical basis for regarding the e^n method as a predictor of the onset of transition, and until this has been achieved, extensions must to some extent be ad hoc. Our view is that the normal modes deduced from the Orr-Sommerfeld equation are not central to the mathematical theory of small disturbances to a shear flow, even granted the parallel flow assumption, and are of little importance physically, as long as they decay in amplitude as x increases. Hence, in any computation of "n", the appropriate starting point should lie on the neutral curve corresponding to the given real value ω_l of ω. We select a point S on this curve where $R = R_S$ and holding the physical frequency constant, i.e., choosing $\omega = \omega_l R_S / R$, allow R to increase. At this stage we abandon the notion that α, β, ϕ are constants and instead allow them to be slowly varying functions of R. This change is generally accepted to be reasonable in two-dimensional studies and needs no further justification here. At each new value of R, it would be reasonable to ask that the appropriate value of ϕ should correspond to a maximum growth rate, as it does at $R = R_S$. This extra condition, that $\alpha_i + \beta_i \tan\phi$ be a minimum regarded as a function of ϕ, completes the specification of α, β at the new value of R. Continuing the procedure, we would then be able to compute a maximum value for N for integrations starting from $\omega = \omega_l$, $R = R_S$ at the value R_T where transition begins. The calculation is now repeated for various values of R_S for given ω_l and then again for various ω_l until an absolute maximum value of N is achieved, which is strictly the n of the e^n method.

With the facilities available to us, this task is of daunting complexity. Therefore, as an initial step we have made two assumptions: 1) that the maximum growth rate at any value of $R > R_S$ is not significantly dependent on ϕ, and 2) for a given value of ω_l, the maximum value for N is achieved when $R_S = R_P$. We recognize the arbitrary nature of the assumption about ϕ and that while for temporal disturbances, the zarf does separate out domains with unstable modes from those with only stable modes. The same cannot be said of this curve for spatial disturbances. Both assumptions need testing in specific flows; Blasius flow over a flat plate being an obvious example to study since the zarf would seem to be defined by $\beta = 0$. From our investigations of this problem (on which we hope to report in detail shortly), it seems that the assumptions of starting the computation on the zarf and taking ϕ constant are not strictly justified, but if a full computation is carried out, the value of n is increased by only a few percent.

In Fig. 3 we display the variation of N with Reynolds number, the two transition lines being that estimated by Gregory et al.[15] and the value 510 at which Chin and Litt[16] found that the ordered vorticity began to break down into turbulence. It is noted in this connection that Theodorsen and

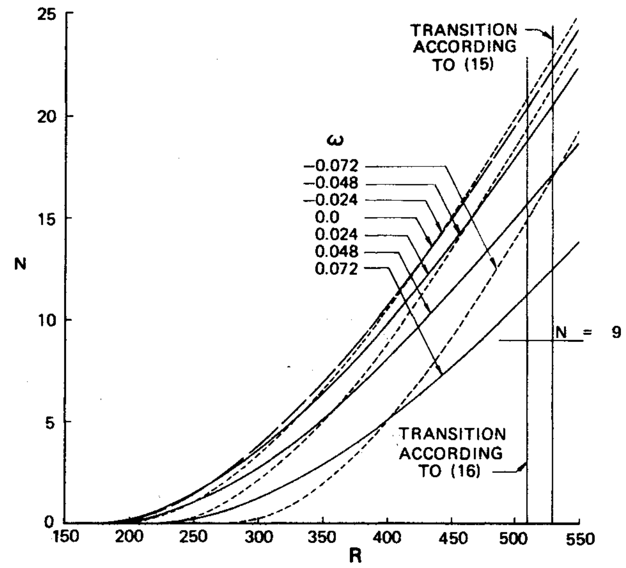

Fig. 3 Variation of N with R for various ω starting at the zarf and holding p constant.

Regier[45] found that the moment coefficient on the rotating disk began to exhibit the rapid rise usually associated with the start of transition for $R>550$, while Cham and Head[46] conclude that the transition processs is complete at $R \simeq 530$.

The most obvious result is that the value of n for transition is much higher than the generally accepted value of 9 for two-dimensional boundary layers and is about 23 if $R_T = 530$ or about 20 if $R_T = 510$. Further, if $n = 9$ where to be insisted on, transition would begin at $R \simeq 400$, whereas the experimentally observed flow has then just begun to exhibit the spiral streaks. It is also worth noting that the shapes of the curves in Fig. 3 are also different from the two-dimensional result since N continues to increase without apparent limit and there is no sign of the formation of an envelope.

Are we then to conclude that the e^n method is useless as a predictor of the transition of three-dimensional boundary layers, since the appropriate value of n is clearly grossly dependent on either the geometry or the physical characteristics of the flow? At the crudest level of interpretation of the success of the method, this is indeed the case, but nevertheless we believe that the results are encouraging and that the method merits further study with the aim of developing a more satisfactory procedure for fixing n than the blanket assumption "$n = 9$."

The experiments of Chin and Litt[16] reveal the existence of a disturbed but ordered laminar flow for $410<R<510$ and indicate that the linearized theory of perturbations has a relevance in this range at least and it may well be that breakdown of the vortices which they observed when $510<R<590$ is susceptible to a causal theory. Gregory et al.[15] observed spiral bands of china clay deposit at ~ -13 deg to the direction of the disk motion. Various attempts to explain this phenomenon have been made by these and subsequent authors; alternatively, it has even been suggested that the spiral bands are not significant for the transition problem. We believe that these spiral bands can be interpreted in the light of our present results. A study of Fig. 3 shows that the disturbances with maximum growth over the whole range of interest have ω very small; but there is very little difference between the marginally unstable case with $\omega_c = 0.017$ and the static case with $\omega = 0$ and having $\alpha = \alpha_0 = 0.345$, $\beta = \beta_0 = 0.060$, $\partial\alpha/\partial\beta = p_0 = 6.98$, and $R = R_0 = 180$. Let us suppose tentatively that the spiral bands are a static phenomenon generated at the appropriate critical Reynolds number. We then see that their phase velocity is inclined at $\simeq 10$ deg to the outward drawn radius of the disk and forward of it. Its group velocity is, on the other hand, (real and) inclined by 8 deg to the backward tangential direction being almost perpendicular to the phase velocity. The number of minima of this marginal disturbance around the disk at $R \simeq 180$ is $2\beta_0 R_0 \simeq 22$; hence, there is a tendency to set up this number of concentrations of china clay. These will be impelled outward along the direction of the group velocity, i.e., virtually along the crests. We assume, and this is probably an oversimplification, that the direction of propagation remains constant so that when the disturbances are observed at $R \sim 400$, we expect to see 22 bands of china clay inclined at an angle -8 deg to the tangential direction of motion of the disk. The observed values of 30 and -13 deg are encouragingly close.

If we had chosen the marginally unstable value R_c of R and the corresponding neutral solution with $\omega_c = 0.017$ instead of R_0 and $\omega = 0$, the wave motions would have been oscillatory tending to smear out the concentrations of china clay. The number of bands set up initially would have been $2\beta_c R_c \simeq 10$ with a direction of propagation also inclined at ~ -8 deg to the tangential motion of the disk.

Thus linear theory seems quite relevant when $R > R_c$ and the correct interpretation of the gross variation of n from the commonly accepted value of 9 is that the linear theory comes to an end through the action of nonlinear terms and these manifest themselves differently in different problems. As explained in the introduction, the Landau constant k is broadly helpful in this regard, even though the theory on which it is based is, at best, only strictly valid at $R = R_c$. Applying this view to the Blasius profile, we see that the important disturbance of the e^n method is supercritically stable,[10] i.e., $k_r < 0$, until quite near the upper branch of the neutral curve when k_r changes sign and the breakdown of the weakly nonlinear theory is then presumably quite rapid. In the supercritical regime, the amplitude of the modal disturbances is probably not fixed only by the linear theory but by a balance between it and the nonlinear forces that are then stabilizing.

A plausible inference is that a parallel situation holds in the flow over a rotating disk apart from the obvious differences in the growth rates predicted by linear theory and the values of R at which $k_r = 0$. Thus, far from being discouraged by the application of the e^n method to this problem, we believe there is cause for optimism that eventually a sound basis for determining n can be found for a wide variety of transition flows.

One final comment—the discrepancy in the shape of Fig. 3 and the corresponding result for two-dimensional boundary layers is not unexpected. It arises because the steady flow here satisfies Rayleigh's criterion for inviscid instability and so the amplification of disturbances is likely to increase without limit as $R \to \infty$; whereas in the other flow, all disturbances have a maximum growth according to linear theory because they are ultimately stable. It is noted that three-dimensional boundary layers are strictly unstable to inviscid crossflow disturbances. Therefore, in such flows we might expect curves for n intermediate to the two discussed here.

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